



Monte Carlo Simulation

Mathematical Foundations, Risk Quantification
and Decision Intelligence

Zainab Jamshed

Founder, CEO & CTO | ZJ Logix

Probability → Simulation → Risk → Decisions

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The Central Question

What should we do when the future is uncertain?

Single-Point Assumption

Demand = 10 units.

Lead time = 2 days.

A single forecast drives the decision.

Distribution-Based View

Demand $\in \{8, 10, 12\}$.

Lead time $\in \{2, 3\}$.

Evaluate decisions across possible futures.

Key insight

The goal is not to predict one perfect future, but to choose decisions that perform well across plausible outcomes.

What Monte Carlo Simulation Does

- ▶ Repeatedly samples uncertain inputs from specified probability distributions.
- ▶ Evaluates a mathematical model for each sampled scenario.
- ▶ Estimates means, event probabilities, dispersion, and tail risks.
- ▶ Compares decisions under uncertainty, but does not itself optimize them.

$$Y = f(x, \xi)$$

x : controllable decision;

ξ : uncertain input;

Y : resulting outcome.

When Is Simulation Appropriate?

Especially Valuable

- ▶ Multiple interacting uncertainties
- ▶ Nonlinear or asymmetric losses
- ▶ Costly adverse or rare events
- ▶ Difficult exact probability calculations
- ▶ Competing operational choices

Potentially Unnecessary

- ▶ Inputs effectively known
- ▶ Simple exact solution available
- ▶ No defensible probability model
- ▶ Computational cost exceeds value

Key insight

The worked inventory example is intentionally small enough for exact independent verification.

Probability: Discrete Uncertain Demand

Daily demand d	$P(D_t = d)$
8	0.25
10	0.50
12	0.25

$$\sum_d P(D_t = d) = 1$$

$$E[X] = \sum_x xP(X = x)$$

$$E[D_t] = 8(.25) + 10(.50) + 12(.25) = 10$$

Key insight

An expected value is a probability-weighted long-run average, not tomorrow's guaranteed demand.

Variance and Standard Deviation

$$\begin{aligned}\text{Var}(X) &= E[(X - E[X])^2] \\ &= E[X^2] - E[X]^2\end{aligned}$$

$$\text{SD}(X) = \sqrt{\text{Var}(X)}$$

For the demand example:

$$\begin{aligned}E[D_t^2] &= 8^2(.25) + 10^2(.50) + 12^2(.25) \\ &= 102\end{aligned}$$

$$\text{Var}(D_t) = 102 - 10^2 = 2$$

$$\boxed{\text{SD}(D_t) = \sqrt{2} \approx 1.41}$$

How a Random Scenario Is Generated

Inverse Cumulative Probability Mapping

Generate:

$$U \sim \text{Uniform}(0, 1)$$

Map the uniform random value to a demand outcome.

Uniform draw	Demand	Probability
$0 \leq U < .25$	8	.25
$.25 \leq U < .75$	10	.50
$.75 \leq U < 1$	12	.25

$$U = .63 \Rightarrow D_t = 10$$

$$U = .91 \Rightarrow D_t = 12$$

The Monte Carlo Estimator

Population Target and Sample Approximation

$$\mu = E[g(X)] = \int g(x)p(x) dx$$

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N g(X_i)$$

- ▶ X_i : sampled input in replication i .
- ▶ $g(X_i)$: resulting model output.
- ▶ N : number of independent, identically distributed trials.
- ▶ $\hat{\mu}_N$: estimated mean output.

Why Repeated Sampling Works

Law of Large Numbers

$$\bar{Y}_N = \frac{1}{N} \sum_{i=1}^N Y_i \xrightarrow[N \rightarrow \infty]{P} E[Y]$$

- ▶ Small N can give noisy estimates; larger N typically improves stability.
- ▶ Larger samples are not guaranteed to be closer on every run.
- ▶ Repetition reduces sampling error, not errors in the model or its assumptions.

Precision, Confidence Intervals and Risk

$$s_Y^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2$$

$$\widehat{SE}(\bar{Y}) = \frac{s_Y}{\sqrt{N}}$$

Approximate 95% confidence interval:

$$\bar{Y} \pm 1.96 \frac{s_Y}{\sqrt{N}}$$

Estimated adverse-event probability:

$$\hat{P}(Y > a) = \frac{1}{N} \sum_{i=1}^N \mathbf{1}\{Y_i > a\}$$

Multiplying N by 4 approximately halves the standard error.

A confidence interval for the mean is not an interval for individual outcomes.

Inventory Case: The Managerial Decision

- ▶ A synthetic FMCG distributor must choose stock S to cover uncertain demand during supplier lead time.
- ▶ Overstock incurs a holding or excess charge.
- ▶ Understock incurs a shortage penalty.
- ▶ One item, one replenishment period, and no backorders or later corrective decisions.
- ▶ All numerical assumptions are teaching examples, not observed company data.

Key insight

This case connects Monte Carlo simulation to inventory decision-making and operations research.

Inventory Case: Uncertain Inputs

Daily Demand D_t

Units	Probability
8	.25
10	.50
12	.25

Lead Time L

Days	Probability
2	.60
3	.40

Assumption:

Daily demands are mutually independent and independent of lead time.

Lead-Time Demand: Expectation and Variance

$$D_L = \sum_{t=1}^L D_t$$

$$E[L] = 2(.6) + 3(.4) = 2.4$$

$$\text{Var}(L) = .24$$

$$E[D_L] = E[L]E[D_t] = 24$$

Variance of a Random Sum

Under the stated independence assumptions:

$$\begin{aligned}\text{Var}(D_L) &= E[L] \text{Var}(D_t) \\ &\quad + \text{Var}(L)(E[D_t])^2 \\ &= (2.4)(2) + (.24)(10^2) = 28.8\end{aligned}$$

$$\text{SD}(D_L) \approx 5.37$$

Stock, Shortage and Economic Loss

Excess inventory:

$$E = (S - D_L)^+$$

Unmet demand:

$$H = (D_L - S)^+$$

where:

$$z^+ = \max(z, 0)$$

Asymmetric Cost Model

$$C(S, D_L) = 2(S - D_L)^+ + 20(D_L - S)^+$$

- ▶ Excess-stock cost: Rs. 2 per unit.
- ▶ Shortage cost: Rs. 20 per unit.

A Complete Simulated Scenario

Decision: $S = 30$ units.

Sample: $L = 3$ days.

Daily demands: (12, 10, 12).

$$D_L = 12 + 10 + 12 = 34$$

$$E = (30 - 34)^+ = 0$$

$$H = (34 - 30)^+ = 4$$

$$\begin{aligned} C(30, 34) &= 2(0) + 20(4) \\ &= \boxed{\text{Rs. } 80} \end{aligned}$$

Key insight

One scenario produces one realization of total demand, excess inventory, shortage and cost.

Five Trials: Illustrating Repetition

Trial	L	Daily demands	D_L	Cost
1	3	12,10,12	34	80
2	2	8,10	18	24
3	3	10,10,10	30	0
4	2	12,12	24	12
5	3	8,8,10	26	8

$$\hat{J}_5(30) = \frac{80 + 24 + 0 + 12 + 8}{5}$$

$$\hat{J}_5(30) = \text{Rs. } 24.80$$

Five trials illustrate the simulation mechanism but are insufficient for dependable managerial estimation.

From Five Trials to Thousands

Estimated Expected Cost for Stock S

$$\hat{J}_N(S) = \frac{1}{N} \sum_{i=1}^N C(S, D_{L,i})$$

Expanding:

$$\hat{J}_N(S) = \frac{1}{N} \sum_{i=1}^N [2(S - D_{L,i})^+ + 20(D_{L,i} - S)^+]$$

- ▶ Generate $N = 1,000, 10,000$ or $100,000$ independent scenarios.
- ▶ Report mean cost, stockout frequency, and standard error.
- ▶ Reuse identical scenarios across candidate stock choices.

Exact Probability: A Benchmark

Consider the event:

$$D_L = 16$$

This requires:

$$L = 2, \quad (D_1, D_2) = (8, 8)$$

Therefore:

$$P(D_L = 16) = .60(.25)(.25)$$

$$P(D_L = 16) = .0375$$

General Discrete Distribution

$$P(D_L = d) = \sum_{l=2}^3 P(L = l) \cdot P\left(\sum_{t=1}^l D_t = d\right)$$

Exact enumeration provides a benchmark against which to verify simulation results.

Simulation Versus Optimization

Simulation: What Can Happen?

For a fixed stock level S , generate and analyze:

$$\{C(S, D_{L,i})\}_{i=1}^N$$

Estimate:

- ▶ Expected cost
- ▶ Stockout probability
- ▶ Risk exposure

Optimization: What Should We Choose?

Search for the best feasible stock decision:

$$S^* = \arg \min_{S \in \mathcal{S}} E[C(S, D_L)]$$

Subject to possible budget, capacity and service constraints.

Key insight

Simulation evaluates uncertain consequences; optimization selects the best feasible decision.

Comparing Stock Decisions: Exact Results

Stock S	Expected Cost (Rs.)	$P(D_L \leq S)$
20	89.90	41.25%
25	46.14	60.62%
30	20.25	86.25%
32	18.20	95.62%
34	20.28	99.38%
40	32.00	100.00%

$$S^* = 32 \text{ units}$$

$$J(S^*) = \text{Rs. } 18.20$$

The exact optimum is 32 units among nonnegative integer stock levels.

Why 32 Units? The Critical-Fractile Rule

Underage and Overage Costs

$$c_u = 20, \quad c_o = 2$$

The critical-fractile condition is:

$$P(D_L \leq S) \geq \frac{c_u}{c_u + c_o}$$

$$\frac{20}{22} = 0.9091$$

Compare cumulative probabilities:

$$P(D_L \leq 30) = 0.8625 < 0.9091$$

$$P(D_L \leq 32) = 0.95625 > 0.9091$$

$$S^* = 32 \text{ units}$$

Independent Check of the Optimum

Expected shortage at $S = 32$:

$$\begin{aligned} E[H] &= (34 - 32)(.0375) \\ &\quad + (36 - 32)(.00625) \\ &= .10 \end{aligned}$$

Expected excess inventory:

$$E[E] = 32 - 24 + .10 = 8.10$$

Expected economic loss:

$$J(32) = 2(8.10) + 20(.10)$$

$J(32) = \text{Rs. } 18.20$

Key insight

A simulated estimate should converge toward this exact cost under the stated probability assumptions.

Service Levels and Downside Risk

A no-shortage service constraint is:

$$P(D_L \leq S) \geq 1 - \alpha$$

Service target	Minimum stock	Cost
95%	32	Rs. 18.20
99%	34	Rs. 20.28
100%	36	Rs. 24.00

VaR measures a high cost quantile. CVaR measures expected cost in the adverse tail.

A risk-aware extension is:

$$\min_{S \in \mathcal{S}} \{E[C] + \lambda \text{CVaR}_{.95}(C)\}$$

Python: Simulation and Decision Comparison

```
import numpy as np

rng = np.random.default_rng(2026)
N = 100_000

L = rng.choice([2, 3], N, p=[.60, .40])

D = rng.choice(
    [8, 10, 12],
    (N, 3),
    p=[.25, .50, .25]
)

DL = np.where(
    L == 2,
    D[:, :2].sum(axis=1),
    D.sum(axis=1)
)

results = []

for S in range(16, 41):
    excess = np.maximum(S-DL, 0)
    shortage = np.maximum(DL-S, 0)

    cost = 2*excess + 20*shortage

    results.append(
        (S, cost.mean(),
         (shortage > 0).mean(),
         cost.std(ddof=1)/np.sqrt(N))
    )

print(min(results, key=lambda r: r[1]))
```

Verification, Validation and Limitations

Internal Checks

- ▶ Sampled frequencies match probabilities
- ▶ $E[D_L] \approx 24$
- ▶ $J(32) \approx 18.20$
- ▶ Nonnegative costs
- ▶ Convergence across replications

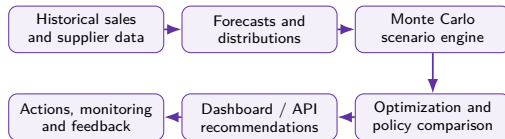
Real-World Checks

- ▶ Historical calibration
- ▶ Holdout testing
- ▶ Dependence and seasonality
- ▶ Rare supplier disruptions
- ▶ Operational capacity
- ▶ Backorders and policy changes

Key insight

More trials reduce Monte Carlo sampling error; they cannot repair biased or unrealistic model inputs.

From Mathematical Model to Decision Platform



Research extensions include multi-period inventory, reorder policies, chance constraints, CVaR and nonanticipative decision-making.

Mathematical Takeaway

$$\xi \sim p(\xi)$$

$$Y = f(x, \xi)$$

$$\hat{J}_N(x) = \frac{1}{N} \sum_{i=1}^N f(x, \xi_i)$$

$$x^* = \arg \min_{x \in \mathcal{X}} E[f(x, \xi)]$$

Uncertainty → **Scenarios** → **Outcomes**
→ **Risk** → **Decisions**